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A Numerical Study of the Hierarchical Ising Model: High-Temperature Versus Epsilon Expansion

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We study numerically the magnetic susceptibility of the hierarchical model with Ising spins ($\sigma = \pm 1$) above the critical temperature and for two values of the epsilon parameter. The integrations are performed exactly using recursive methods which exploit the symmetries of the model. Lattices with up to 2^{18} sites have been used. Surprisingly, the numerical data can be fitted very well with a simple power law of the form $(1-\beta/\beta_0)^{-g}$ for the *whole* temperature range considered. This approximate law implies a simple approximate formula for the coefficients of the high-temperature expansion, and, more importantly, approximate relations among the coefficients themselves. We found that some of these approximate relations hold with errors less then 2%. On the other hand, g differs significantly from the critical exponent γ calculated with the epsilon expansion, even when the fit is restricted to intervals closer to β_c. We discuss this discrepancy in the context the renormalization group analysis of the hierarchical model.

KEY WORDS: Renormalization group; critical exponents; hierarchical models; high-temperature expansion; Ising models; epsilon expansion.

1. INTRODUCTION

The renormalization group (RG) method[1] is a powerful tool to handle critical phenomena and to approach the continuum limit of lattice models. However, its practical implementation usually requires approximations. In his original paper, Wilson made order-of-magnitude estimates of various terms contributing to the partition function of the Landau–Ginzburg model and derived the so-called approximate recursion formula.[1] In this approximation, the RG transformation is reduced to a single integral equation which can be studied using numerical methods or functional analysis.

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Recursion formulas closely related to the approximate recursion formula hold *exactly* for the hierarchical models.[2] This is due to the large group of symmetries of the Hamiltonians of these models. The RG transformation for these models has been studied in great detail and rigorous results concerning the epsilon expansion of the critical exponents are available in the literature.[3-5]

The fact that the RG transformation can be handled easily for hierarchical models suggests the use of these models as an approximation[6] for nearest-neighbor models. The main technical problems in proceeding this way are how to derive explicitly the approximate models and how to improve systematically the approximation. Recently, one of us[7] has answered these questions for the Gaussian models where everything can be calculated explicitly. In order to extend this method to interacting models, one should be able to calculate the average value of perturbation terms added to the hierarchical Hamiltonian. For Ising models, where the spin σ takes only the values ± 1, this task can be carried out numerically in an efficient way. It has also been suggested[8] that such an approach might shed some light on Polyakov's conjecture[9] for the 3D Ising model. In preparation for these calculations, we first checked the agreement between the numerical calculations for the (unperturbed) hierarchical Ising model and analytical results. In doing so, we found surprising results which are reported in the following.

In this paper, we calculate numerically the magnetic susceptibility per site—the susceptibility for short—of the hierarchical Ising model as a function of the temperature and for two values (0 and 1) of ε, the parameter used in the ε-expansion. Calculations have been carried with up to 2^{18} sites. The numerical integration made use of the symmetries of the model in order to cut down the time of computation logarithmically. However, no approximations have been made and the numbers shown below are exact up to roundoff errors. These errors were analyzed by changing from simple to double precision. In all the cases considered, this only affected the fifth significant digit of the susceptibility in the worse cases. Our calculations have been mostly restricted to the high-temperature region and its boundary. In other words, the parameter β, proportional to the inverse temperature, will run between 0 and a critical value β_c. However, at the beginning, a few calculations will be made in the low-temperature region in order to locate β_c.

Surprisingly, we found that the numerical data can be fitted very precisely with a simple power law of the form $(1-\beta/\beta_0)^{-g}$ in the *whole* high-temperature region, i.e., for $\beta \in [0, \beta_c)$. As a consequence, it is possible to obtain a simple approximate formula for the high-temperature coefficients of the susceptibility in terms of g and β_0. This approximate formula

implies approximate relations among the coefficients themselves. We have checked analytically that in the infinite-volume limit one of these relations stays approximate with relative errors on the order of 2%. On the other hand, the values of g obtained numerically differ significantly from the values of the critical exponent γ obtained in the ε-expansion. This discrepancy slightly diminishes, but remains significant when fits are performed over an appropriate subinterval of the form $[\beta_{\min}, \beta_c)$. This means that for one reason or another, we never get close enough to Wilson's fixed point where the ε-expansion is valid. The most plausible explanation suggested by the RG analysis is that near β_c, the approach of the fixed point is very slow and 16 or 17 iterations are not enough to obtain accurate information concerning the critical behavior. An alternative explanation could be the existence of another fixed point distinct from Wilson's fixed point. However, we are not aware of any result favoring such an outcome.

This paper is organized as follows. In Section 2, we introduce the hierarchical model and the numerical method of integration. In Section 3, we check the scaling laws for the variance of the total spin and we determine β_c. In Section 4, we analyze the temperature dependence of the susceptibility and we show that the data can be fitted very well with a simple power law. We compare these results with the high-temperature expansion (Section 5) and the ε-expansion (Section 6). Finally, we discuss our present understanding of the results in the conclusions.

2. THE HIERARCHICAL ISING MODEL

In this section we describe the hierarchical Ising model and the basic ideas of the numerical calculation performed. Hierarchical models[2] are specified by a nonlocal Hamiltonian bilinear in the spin variables and a local measure of integration. We consider here the case of an Ising measure, where the spins take only the values ± 1. The nonlocal Hamiltonian transforms in a simple way under an RG transformation and the transformation only affects the measure of integration.

In order to facilitate reading of the rest of this paper, let us first recall the form of the nonlocal Hamiltonian of the hierarchical model and some of the remarkable properties associated with it. The main purpose of this paragraph is to motivate Eqs. (2.1) and (2.4). For convenience, we label the sites with n indices $i_n, \ldots, i_1$, each index being 1 or 2. In order to visualize the meaning of this notation, one can divide the 2^n sites into two boxes, each containing 2^{n-1} sites. If $i_n = 1$, the site is in the first box; if $i_n = 2$, the site is in the second box. Repeating this procedure n times (for the two boxes, their respective two subboxes, etc.), we obtain an unambiguous

labeling for each of the sites. Two sites differing only by i_1 are in the same box of size 2. The hierarchical Hamiltonian, written in Eq. (2.1) below, depends only on $S_{i_n,\ldots,i_2} = \sigma_{i_n,\ldots,i_2,1} + \sigma_{i_n,\ldots,i_2,2}$ and plays no rule when we integrate over the spins while keeping the $S_{i_n,\ldots,i_2}$ constant. After performing this integration, we consider $S_{i_n,\ldots,i_2}$ as a new local variable and we reabsorb the local part (in the new variable) of the Hamiltonian into the new local measure [see Eq. (2.4) below]. The procedure can be repeated and summarized with a simple recursion formula provided that we couple all the spins located at sites within a box of size 2^l with the same strength $(c/4)^l$. This property characterizes the Hamiltonian of a hierarchical model with 2^n sites, which reads

$$H = -\frac{1}{2}\sum_{l=1}^{n}\left(\frac{c}{4}\right)^l \sum_{i_n,\ldots,i_{l+1}} \left(\sum_{i_l,\ldots,i_1} \sigma_{(i_n,\ldots,i_1)}\right)^2 \tag{2.1}$$

The model has a free parameter c for which we shall use the parametrization

$$c = 2^{1-2/D} \tag{2.2}$$

The parameter of the epsilon expansion will be defined as

$$\varepsilon = 4 - D \tag{2.3}$$

This choice has been justified in ref. 8, but different conventions exist in the literature. The fact that the nontrivial fixed point of the RG transformation merges into the Gaussian fixed point when D reaches 4 from below is explained at length in refs. 3, 5, and 6. On the other hand, the model has no second-order phase transition for $D \leqslant 2$.[2] These two results can be understood heuristically from the result[10] that $2/D$ is the Hausdorff dimension of the random walk associated with the kernel of the Hamiltonian (2.1). When $D > 4$, the Hausdorff dimension is less than one-half of the dimension of the space and self-intersections of random paths is unlikely. When $D < 2$, the Hausdorff dimension is larger than the dimension of the space and the model has only a high-temperature phase. In the actual calculations reported below, we have selected the values 0 and 1 for ε.

We are interested in calculating the magnetic susceptibility of the hierarchical Ising model. This quantity can be calculated easily if we know the probability for the total spin, denoted $P_n(S)$. This probability is obviously β-dependent, even though we shall not write it explicitly. This probability will be calculated recursively using the RG method without a rescaling of the spins. We first integrate the spins inside boxes of size 2, keeping the sum of the spins in each box constant. We then include the

terms with $l=1$ in (2.1) in a new local measure for these sums. We repeat this procedure n times and obtain a measure for the total spin which can be normalized as a probability. Note that a choice of indices $i_n,\ldots, i_{l+1}$ completely specifies a box of size 2^l with the subdivision described above. Generalizing the notation used at the beginning of this section, we call the sum of the spin inside this box $S_{i_n,\ldots,i_{l+1}}$. It can take all the even values between -2^l and 2^l. Obviously, $S_{i_n,\ldots,i_{l+1}} = S_{i_n,\ldots,i_{l+1},1} + S_{i_n,\ldots,i_{l+1},2}$. With these notations, the recursion formula reads

$$P_{l+1}(S_{i_n,\ldots,i_{l+1}}) = C_{l+1} \operatorname{Exp}\left[\frac{1}{2}\beta\left(\frac{c}{4}\right)^{l+1}(S_{i_n,\ldots,i_{l+1}})^2\right] \times \sum_{S_{i_n,\ldots,i_{l+1}} = S_{i_n,\ldots,i_{l+1},1} + S_{i_n,\ldots,i_{l+1},2}} P_l(S_{i_n,\ldots,i_{l+1},1})\, P_l(S_{i_n,\ldots,i_{l+1},2}) \tag{2.4}$$

The constant C_{l+1} is adjusted in such a way that the sum of the probabilities is 1. Strictly speaking, it is not necessary to impose such a normalization during the intermediate steps of the calculation; however, it keeps the numbers reasonably small.

Note also that the spin variables are not rescaled in Eq. (2.4) because we are interested in calculating average values involving the original spin variables. However, it is straightforward to reabsorb each time a factor $(c/4)^{1/2}$ in the spin variables and, at the same time, to obtain a regular RG transformation from Eq. (2.4). It is important to notice that if we drop the $l=1$ part of the Hamiltonian (2.1), which is local in the new variable $S_{i_n,\ldots,i_2}$, and if we reabsorb a factor $(c/4)^{1/2}$ in $S_{i_n\ldots i_2}$, then the Hamiltonian is invariant (in the thermodynamic limit). This property will be used in the next section.

The recursion formula (2.4) has been implemented with a computer program. In order to calculate $P_n(S)$, we have to repeat 2^n times a calculation involving roughly 2^n operations. Consequently, the time necessary to calculate $P_n(S)$ scales approximately like 4^n. With the fastest computer at our disposal, a DEC alpha 3000/400, it takes about 10 min to calculate $P_{17}(S)$ from $P_{16}(S)$ when programmed with simple precision in FORTRAN.

Due to the size of the calculation, it is clearly necessary to check for roundoff errors. We have studied the size of these errors by repeating the calculation with double precision instead of simple precision for a large sample of values of β. For n up to 16, we have found very good agreement between the two calculations, the differences showing up in the sixth significant digit of the susceptibility. Differences in the fifth digits were observed

for $n = 17$ and 18. Calculations for $n = 19$ and beyond have shown less stability, require a lot of computer time, and will not be reported here.

The main quantity of interest for us is the average of the square of the total spin, denoted $X_n(\beta)$ and defined as

$$X_n(\beta) = \sum_S P_n(S) S^2 \tag{2.5}$$

Above the critical temperature, this quantity divided by the number of sites has a finite thermodynamic limit. It will be called the magnetic susceptibility per site and denoted $\chi_n(\beta)$,

$$\chi_n(\beta) = \frac{X_n(\beta)}{2^n} \tag{2.6}$$

3. THE SCALING LAWS AND THE DETERMINATION OF β_c

The RG method has definite predictions[5, 11] for the large-n behavior of X_n. In the following we shall recall the main results and show that our numerical calculations reproduce these results with good precision. The case $D = 3$ will be discussed in complete detail, while the results for $D = 4$ will only be briefly commented upon.

It is convenient to express X_n as a power of the square of the number of sites, namely

$$X_n(\beta) = 2^{2n\omega(\beta, n)} \tag{3.1}$$

We could have included a constant of proportionality in this definition; however, the practical advantage of Eq. (3.1) is that $\omega(\beta, n)$ can be obtained immediately from the numerical value of $X_n(\beta)$. The disadvantage of this definition is that the n dependence of $\omega(\beta, n)$ is more important than if we had introduced a proportionality constant. However, we shall ultimately consider the ratios $X_{n+1}(\beta)/X_n(\beta)$ and the determination of a constant of proportionality can be bypassed.

The theoretical value of $\omega(\beta, n)$ can be easily estimated for large or small values of β. For large values of β, all the spins tend to align and $\omega(\beta, n)$ gets close to 1. For small values of β, the spins at different sites become uncorrelated and the variance of the total spin can be approximated by the sum of the individual variances. In other words, $X_n(\beta)$ scales like the number of sites and $\omega(\beta, n)$ gets close to 1/2. For large n, $\omega(\beta, n)$ is attracted by 1 if β exceeds a critical value denoted β_c (low-temperature regime), and by 1/2 if β is less than β_c (high-temperature regime). When β is exactly β_c, $\omega(\beta, n)$ tends to a value between 1 and 1/2

called ω_c. In this case, for n large enough, the local measure tends to a nontrivial fixed point. Remarkably, the value of ω_c can be calculated exactly.[5] This can be derived from the fact discussed in the previous section that if we reabsorb a factor $(c/4)^{1/2} = 2^{-(1/2+1/D)}$ in the new spin variable the new Hamiltonian is the same as the initial Hamiltonian (in the thermodynamic limit). This together with the fact that the measure approaches a fixed point implies that $X_{n+1}(\beta_c) = (4/c)\, X_n(\beta_c)$. Comparing with (3.1), we obtain $\omega_c = 1/2 + 1/D$.

We shall now illustrate these results and estimate β_c for $D = 3$. In order to get a rough idea of the value of β_c, we have plotted in Fig. 1 the trajectories $\omega(\beta, n)$ from $n = 1$ to $n = 16$ and for $\beta = 0.1, 0.2, \ldots, 1.6$. It appears clearly that the separation between the two domains of attraction occurs for a value of β between 1.1 and 1.2. We have repeated this calculation for $\beta = 1.10, 1.11, \ldots, 1.20$ as shown in Fig. 2. This restricts β_c to the interval [1.17, 1.19]. At this point, it is more informative to consider ratios of susceptibilities at two successive values of n, because this quantity is independent of the constant of proportionality which we could have introduced in Eq. (3.1). Remembering that χ_n denotes the susceptibility *per site* [see Eq. (2.6)] and using Eq. (3.1) with ω_c, we obtain

$$\lim_{n \to \infty} \mathrm{Log}_2 \left(\frac{\chi_{n+1}(\beta_c)}{\chi_n(\beta_c)} \right) = \frac{2}{D} \tag{3.2}$$

Figure 3 displays $\mathrm{Log}_2(\chi_{n+1}(\beta)/\chi_n(\beta))$ for three values of β and for n up to 17. For $\beta = 1.179$, this quantity stays within 1% of the critical value 2/3 for the last six iterations. If β is increased or decreased by 0.001, the same

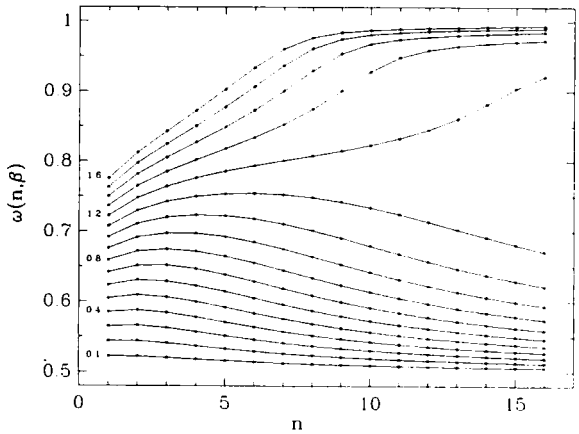

Fig. 1. $\omega(n, \beta)$ versus n for $D = 3$ and values of β going from 0.1 to 1.6 by steps of 0.1.

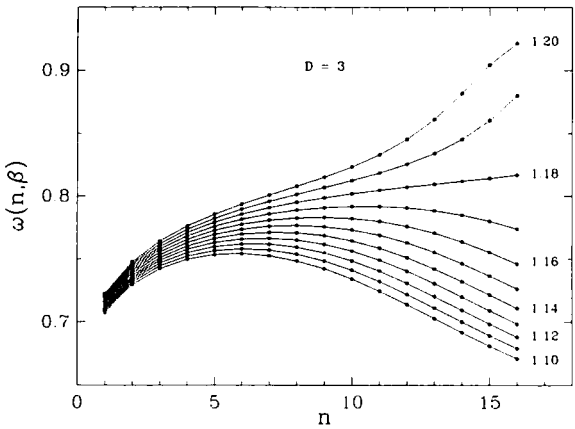

Fig. 2. $\omega(n, \beta)$ versus n for $D = 3$ and values of β going from 1.1 to 1.2 by steps of 0.01.

quantity departs by about 10% from 2/3 when $n = 17$. From this, we conclude that 1.179 approximates β_c with a precision better than 0.001. It is important to notice that this determination does not depend on the maximal value of n used, since the bifurcation will persist for larger n.

Applying the same procedure for $D = 4$, we found that β_c should be inside the interval [0.66, 0.67]. Figure 4 shows that 0.665 approximates β_c with a precision better than 0.001.

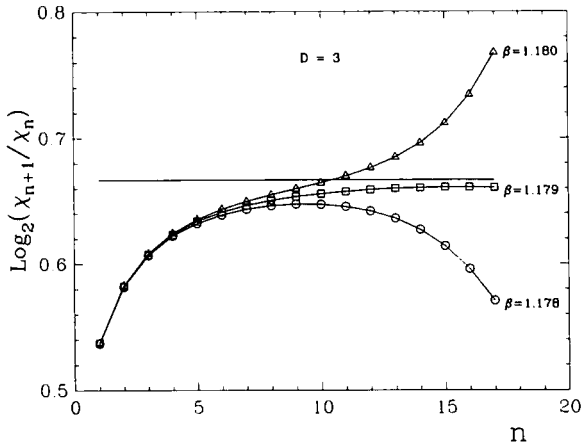

Fig. 3. Log_2 of the ratio of two successive values of the susceptibility for $D = 3$ and for $\beta = 1.178$, 1.179, and 1.180.

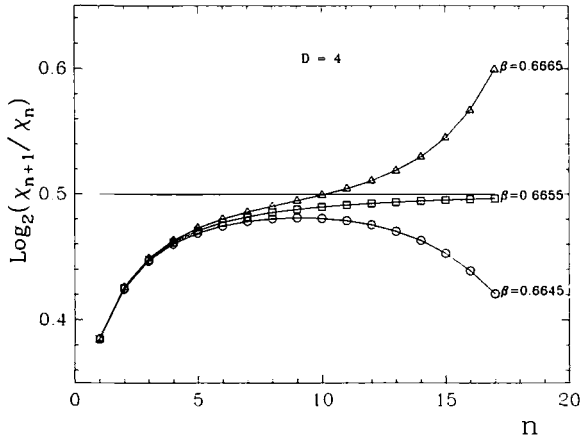

Fig. 4. Log_2 of the ratio of two successive values of the susceptibility for $D=4$ and for $\beta=0.6645$, 0.6655, and 0.6665.

4. FITTING THE NUMERICAL DATA WITH A SIMPLE POWER LAW

In this section, we report our numerical results concerning the β dependence of the magnetic susceptibility per site at $D=3$ and $D=4$ and for $\beta<\beta_c$. Again, we start with the case $D=3$ and discuss it in detail, while the case $D=4$ will be presented more rapidly later in this section. The interpretation of these results is discussed in the next sections.

We have calculated $\chi_n(\beta)$ for $D=3$ and n up to 16 and for values of β between 0 and 1.18 separated by intervals of length 0.01. As expected, the susceptibility rises sharply when β gets close to 1.18. We have displayed the results for $n=16$ and $\beta\leqslant 1.1$ in Fig. 5. Results for $n=14$ or $n=15$ would have been hardly distinguishable from $n=16$ on a graph of this size. On the other hand, the n dependence becomes more notable when β is closer to its critical value, as shown in Fig. 6, which also includes results for $n=17$. Except for the difference of scales, the resemblance between Figs. 5 and 6 is striking. This can be understood from the fact that the numerical data can be fitted very precisely with a simple power law of the form

$$\chi_n(\beta)=(1-\beta/\beta_0)^{-g} \tag{4.1}$$

in the *whole* interval $[0, \beta_c)$. In order to justify this claim, we first notice that (4.1) implies the inverse logarithmic derivative of the susceptibility is a linear function, namely

$$\left\{\frac{d}{d\beta}\,\mathrm{Log}[\chi_n(\beta)]\right\}^{-1}=(g)^{-1}(\beta_0-\beta) \tag{4.2}$$

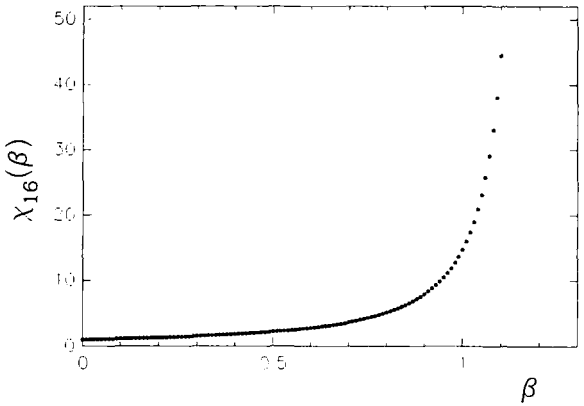

Fig. 5. The magnetic susceptibility for $D = 3$, $n = 16$, and values of β going from 0 to 1.10 by steps of 0.01.

We now approximate this inverse logarithmic derivative by

$$\frac{\Delta\beta}{\Delta \operatorname{Log}[\chi_n(\beta+\Delta\beta/2)]} = \frac{\Delta\beta}{\operatorname{Log}[\chi_n(\beta+\Delta\beta)] - \operatorname{Log}[\chi_n(\beta)]} \tag{4.3}$$

with $\Delta\beta = 0.01$, the interval used here. This function is plotted in Fig. 7 for $n = 16$. Remarkably, the numerical data are barely distinguishable from the least square linear fit

$$\Delta\beta/\Delta \operatorname{Log}[\chi_{16}(\beta)] = 0.80839 - 0.67843\beta \tag{4.4}$$

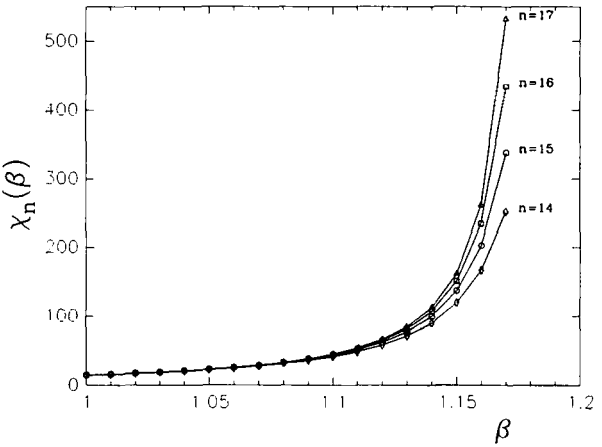

Fig. 6. The magnetic susceptibility for $D = 3$, $n = 14$, 15, 16, and 17, and values of β going from 1.00 to 1.17 by steps of 0.01.

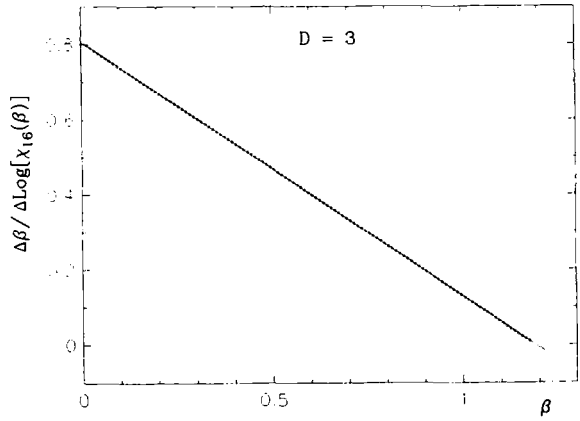

Fig. 7. A discrete version of the inverse logarithmic derivative of the susceptibility and its linear fit for $D = 3$ and $n = 16$.

Identifying Eqs. (4.2) and (4.4) yields $g = 1.4740$ and $\beta_0 = 1.1916$. The value of β_0 is close to the volume-independent result $\beta_c = 1.179$ obtained in the previous section. The value of g obtained above will be compared with analytical results in the next two sections.

The graphs for $n = 14$ or 15 look almost identical. The linear fits are, respectively,

$$\begin{aligned} \Delta\beta/\Delta \ \mathrm{Log}[\chi_{15}(\beta)] &= 0.80849 - 0.67712\beta \\ \Delta\beta/\Delta \ \mathrm{Log}[\chi_{14}(\beta)] &= 0.80867 - 0.67508\beta \end{aligned} \tag{4.5}$$

For those who would attempt to reproduce these results with five significant figures, we mention that the fits have been done without including 1.18, which is clearly above β_c. The change in the fit when the interval is restricted to a subinterval is discussed in detail in the next two sections.

In order to see the corrections to the linear behavior, we have plotted in Fig. 8 the difference between the linear fit and the data, denoted $E(\beta)$, for $n = 14$, 15, and 16. These difference are not larger than 0.003 in absolute values and have interesting regularities which will be discussed at the end of this section.

We have intentionally used simple-precision data to plot Fig. 8 in order to give an idea of the roundoff errors of the method. Small irregularities are visible especially in the low-β region, where their typical size is 10^{-4}. On the other hand, $E(\beta)$ is smoother for $\beta > 0.8$. The size of these numerical errors is compatible with the claim made in Section 2 that numerical errors affect only the sixth significant digit of the susceptibility.

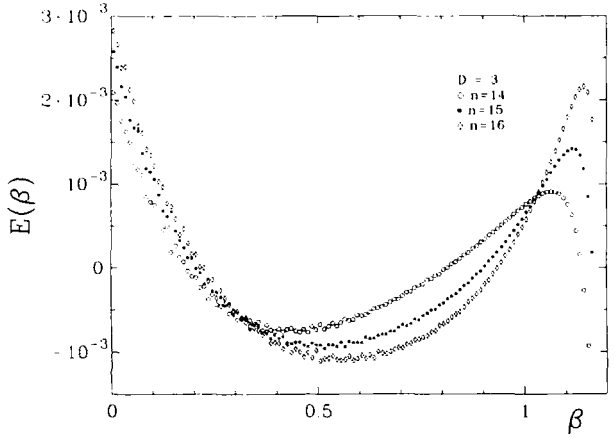

Fig. 8. Difference between the discrete version of the inverse logarithmic derivative of the susceptibility and its linear fit for $D = 3$ and $n = 14$, 15, and 16.

Indeed, if one of the χ_n is replaced by $(1+\delta)\chi_n$ in Eq. (4.3), this creates an error $\delta\, \Delta\beta/[\Delta \operatorname{Log}(\chi)]^2$. A simple inspection shows that δ can be amplified by a factor of order 100 if β is not too close to β_c.

We have also considered the calculation of the quantity in Eq. (4.3) with smaller values of $\Delta\beta$, namely 10^{-3} and 10^{-4}. This study yielded no new information in particular in small regions near β_c.

A similar procedure has been followed for $D = 4$. In Fig. 9 we display the quantity $\Delta\beta/\Delta \operatorname{Log}[\chi_{16}(\beta)]$. Again the departures from linearity are

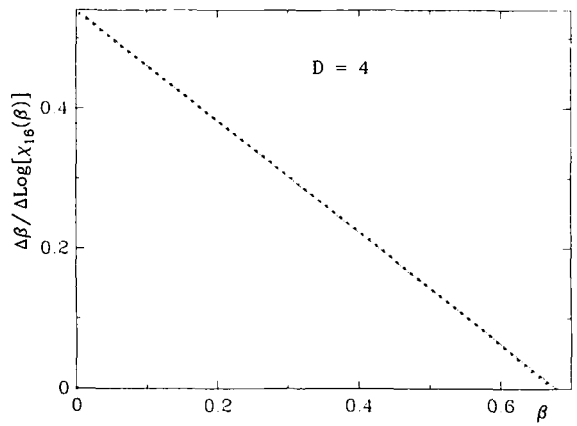

Fig. 9. A discrete version of the inverse logarithmic derivative of the susceptibility and its linear fit for $D = 4$ and $n = 16$.

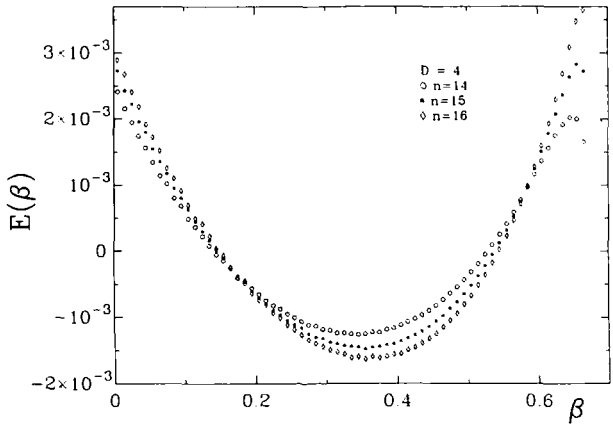

Fig. 10. Difference between the discrete version of the inverse logarithmic derivative of the susceptibility and its linear fit for $D = 4$ and $n = 14$, 15, and 16.

small and the changes in the coefficients of the linear fit for $n = 14$ and 15 are of the same order as those for $D = 3$. The difference between the linear fit and the data for $n = 14$, 15, and 16 is displayed in Fig. 10. For further comparison, a fit over the whole interval for $n = 16$ yields $g = 1.2568$ and $\beta_0 = 0.6803$.

Figures 8 and 10 show that the differences between the fit and the data are larger near 0 and β_c. In both regions, the data are below the fit in Figs. 7 and 9. In the high-temperature region, the fit (made over the whole interval), overestimates the absolute value of the slope of the data. In the critical region, the situation is more complicated and will be discussed in more detail in Section 6. Figures 8 and 10 also show that in both the high-temperature and the critical region the differences between the fit and the data increase slightly with n.

5. COMPARISON WITH THE HIGH-TEMPERATURE EXPANSION

Up to now, we have shown that at finite, but by no means small, volume, the power behavior of the magnetic susceptibility is very similar in the high-temperature region and in the critical region. The fact that the discrepancy between Eq. (4.1) and the data is small but significantly nonzero shows that Eq. (4.1) is in general not exact at finite volume. At the end of this section, we shall show that Eq. (4.1) is not exact in the infinite-volume limit, by calculating three terms in the high-temperature expansion of the susceptibility. This means that, despite the remarkable linear behavior

appearing in Figs. 8 and 10, there are no special reasons to use the whole interval $[0, \beta_c)$ in order to obtain precise results concerning the high-temperature behavior of the susceptibility.

The high-temperature expansion of the susceptibility reads

$$\chi_n(\beta) = 1 + b_{(1,n)}\beta + b_{(2,n)}\beta^2 + \cdots \tag{5.1}$$

Despite the nonlocality of the model, the two first coefficients can be written in terms of truncated geometrical series. A straightforward but tedious calculation yields

$$b_{(1,n)} = \left(1-\frac{c}{4}\right)^{-1}\left\{\frac{c}{4}\left[1-\left(\frac{c}{2}\right)^n\right]\left(1-\frac{c}{2}\right)^{-1} - (2^n-1)\left(\frac{c}{4}\right)^{n+1}\right\} \tag{5.2a}$$

and

$$b_{(2,n)} = (b_{(1,n)})^2 - \left(1-\frac{c}{4}\right)^{-2}\left\{\left(\frac{c}{4}\right)^2\left(1-\left(\frac{c^2}{8}\right)^n\right)\left(1-\frac{c^2}{8}\right)^{-1} - 2\left(\frac{c}{4}\right)^{n+2}\left[1-\left(\frac{c}{2}\right)^n\right]\left(1-\frac{c}{2}\right)^{-1} - (2^n-1)\left(\frac{c}{4}\right)^{2(n+1)}\right\} \tag{5.2b}$$

On the other hand, the parametrization of the susceptibility (4.1) has the high-temperature expansion

$$\left(1-\frac{\beta}{\beta_0}\right)^{-g} = 1 + \frac{g}{\beta_0}\beta + \frac{g(g+1)}{2\beta_0^2}\beta^2 + \cdots \tag{5.3}$$

We now compare the two first coefficients of Eqs. (5.1) and (5.3) in the case $n = 16$ and $D = 3$. If we use the values of g and β_0 obtained from a fit over the whole interval given in Eq. (4.4), we obtain $g/\beta_0 = 1.2370$, while from Eq. (5.2) we obtain $b_{(1,16)} = 1.2415551$. Similarly for the second coefficient, we obtain $g(g+1)/2\beta_0^2 = 1.2842$ and $b_{(2,16)} = 1.2776946$. In both cases, the numbers agree within less than 0.01. As one could expect, a better agreement with the high-temperature expansion can be reached by restricting the fit to the high-temperature region. For instance, if we use the interval $[0, 0.1]$ to perform the fit, we obtain $\beta_0 = 1.21705$ and $g = 1.51091$, which implies $g/\beta_0 = 1.2415$ and $g(g+1)/2\beta_0^2 = 1.2806$.

On the other hand, it is also possible to obtain approximate values for g and β_0 by comparing the two first terms of Eqs. (5.1) and (5.3). Solving $g/\beta_0 = b_{(1,n)}$ together with $g(g+1)/2\beta_0^2 = b_{(2,n)}$, we obtain

$$\begin{aligned} g &= \left(\frac{2b_{(2,n)}}{b_{(1,n)}^2} - 1\right)^{-1} \\ \beta_0 &= \left(\frac{2b_{(2,n)}}{b_{(1,n)}} - b_{1,n}\right)^{-1} \end{aligned} \tag{5.4}$$

In the case $n = 16$, $D = 3$, we obtain $g = 1.5203$ and $\beta_0 = 1.2245$, which differ by about 0.01 from the corresponding values obtained from the restricted fit and by less than 0.05 from one obtained from the whole-interval fit. The main interest of this approach is that we can prove that Eq. (4.1) is not exact even for arbitrarily large n. For Eq. (4.1) to be exact, we need all the terms of Eqs. (5.1) and (5.3) to match. Equation (5.4) guarantees the matching of the first two terms. At third order the matching requires $g(g+1)(g+2)/6\beta_0^3 = b_{(3,n)}$. Plugging the values of g and β_0 obtained in Eq. (5.4), we obtain the relation

$$b_{(3,n)} = \frac{b_{(2,n)}}{3}\left(\frac{4b_{(2,n)}}{b_{(1,n)}} - b_{(1,n)}\right) \tag{5.5}$$

Due to the nonlocality of the model, the calculation of $b_{(3,n)}$ has a difficulty comparable with the calculation of a one-loop Feynman diagram. The methods used will be reported elsewhere.[15] The results has been checked with two independent methods. The main result is that Eq. (5.5) is not exact. The difference between the value of $b_{(3,n)}$ given by Eq. (5.4) and the actual value is small, decreases with n, but finally stabilizes at -0.02122 for $n > 30$. It is nevertheless remarkable that approximate relations among the high-temperature coefficients can be found. The deep reason for these approximate relations remains to be understood.

We have performed a similar analysis in the case $D = 4$ and observed essentially the same features as in the case $D = 3$. In particular, we obtain that the actual value of $b_{(3,n)}$ is 4.8517065, while the value obtained from Eq. (5.5) is 4.797958. The difference between the two values is less than 2% of $b_{3,n}$.

In conclusion, our numerical calculation has shed new light on the high-temperature expansion of the hierarchical model. It reveals approximate relations among the coefficients. We also found that Eq. (4.1) is not exact in the infinite-volume limit and consequently cannot be used to obtain a precise estimate of the critical exponent γ. The conventional high-temperature calculation of the critical exponents[12] requires a larger number of coefficients than what is presently available to us. In the case of the critical exponents of the nearest-neighbor Ising models, small discrepancies between high-temperature and RG calculations have been noticed in the past (see ref. 13 for references to the vast literature on this subject). Several authors[14] have shown that these discrepancies can be removed by an appropriate treatment of the confluent singularities. We are planning to investigate this issue in the present case.

6. COMPARISON WITH THE EPSILON EXPANSION

The critical exponents of the hierarchical model near Wilson's non-trivial fixed point can be calculated using the ε-expansion. Near this fixed point, the linearized RG transformation is dominated by its two largest eigenvalues, λ_1 and λ_2. The critical index γ depends only on λ_1 and reads

$$\gamma = \frac{\mathrm{Log}(2/c)}{\mathrm{Log}(\lambda_1)} \tag{6.1}$$

This definition has been carefully justified in refs. 3 and 5. We give here a quick explanation. The susceptibility per site with 2^n sites is equal to $2/c$ (see Section 3) times the susceptibility per sites calculated with 2^{n-1} sites but with $\beta - \beta_c$ multiplied by λ_1 (this follows from a more detailed argument). This implies that $\lambda_1^\gamma = 2/c$, which is equivalent to Eq. (6.1). Instead, if $\beta = \beta_c$, the susceptibility per sites is multiplied by $2/c = 2^{2/D}$ with no change in temperature and we obtain the critical scaling law (3.2).

With the convention of Eq. (2.2), one obtains at first order in ε

$$\begin{aligned} \lambda_1 &= \sqrt{2} + \varepsilon \frac{(\log 2)\sqrt{2}}{24} \\ \lambda_2 &= 1 - \varepsilon \frac{(\log 2)}{4} \end{aligned} \tag{6.2}$$

In the case $D = 4$, the exact values are obtained by setting $\varepsilon = 0$ in the above equations, which implies $\gamma = 1$. In the case $D = 3$, more accurate calculations are available in the literature. An elaborate resummation method[4] yields $\lambda_1 = 1.427$, which implies $\gamma = 1.300$. This numerical value agrees with the results obtained from a numerical analysis of the recursion formula given by Bleher in the second appendix of ref. 3. In addition, this numerical analysis provides the approximate value $\lambda_2 = 0.85$.

It is clear that the values $g = 1.474$ (1.257 resp.) obtained from a whole interval fit in the case $D = 3$ ($D = 4$ resp.), $n = 16$ differs significantly from $\gamma = 1.300$ (1.000 resp.). However, as explained in the previous section, Eq. (4.1) is only an approximate formula and despite the good-looking linear behavior shown by Figs. 7 and 9, there are no special reasons to use the whole interval to obtain precise values characterizing the critical behavior. We have thus repeated the fitting procedure over intervals $[\beta_{\min}, \beta_c)$ varying $\beta_{\min}$ from 0 to values close to β_c. The results are displayed in Fig. 11 for $D = 3$ and Fig. 12 for $D = 4$. These graphs show that by reducing the interval where the linear fit is performed, we can obtain values of g which are closer to the value of γ predicted by the ε-expansion. For instance, in the case $D = 3$ and $n = 16$, if we only consider the interval

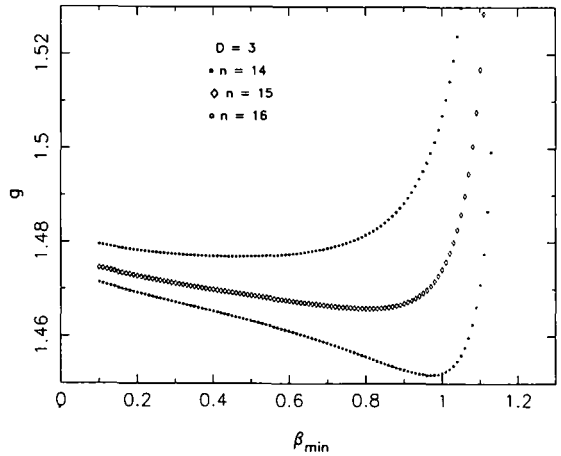

Fig. 11. Numerical estimation of g for $D=3$ and n between 14 and 16, when the fit is restricted to the interval $[\beta_{min}, \beta_c)$.

[0.97, 1.17], we obtain $g=1.4518$. In the case $D=3$ and $n=17$, we obtain $g=1.4370$ for the interval [1.00, 1.17]. These values of g differ more significantly from the typical value $g=1.51$ obtained in the high-temperature region but are still far away from 1.300. Importantly, we see that if we adjust the interval in order to get the lowest possible value of g, we obtain values which decrease much more rapidly with n than the values obtained

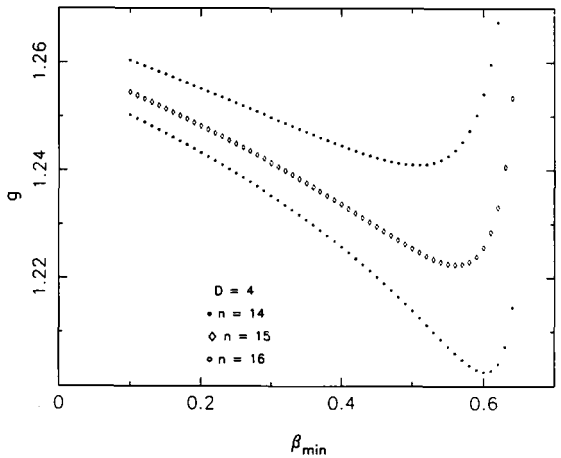

Fig. 12. Numerical estimation of g for $D=4$ and n between 14 and 16, when the fit is restricted to the interval $[\beta_{min}, \beta_c)$.

from a whole-interval fit, the latter being "stabilized" by the high-temperature region.

On the other hand, if we consider intervals closer and closer to β_c, the value of g starts rising. This comes from the fact that at finite volume, it is impossible to obtain a true singularity. Consequently, the absolute value of the slope starts decreasing when β gets too close to β_c. From Eq. (4.2), this means that g starts increasing. This effect is barely visible in Figs. 7 and 9, but becomes more obvious if we plot a few points above β_c. This indicates that even though we have used lattices with up to $2^{17} = 131{,}072$ sites, we do not probe accurately the critical region.

The possibility that a larger number of iterations is necessary to obtain accurate results concerning the critical behavior is supported by the RG analysis. The fact that the second eigenvalue λ_2 is close to 1 (0.85, see above) in the case $D = 3$ suggests that the approach of the nontrivial fixed point or more generally of the unstable manifold can be rather slow $[(0.85)^{17} = 0.063]$. The situation is worse in the case $D = 4$, where λ_2 is exactly 1.

It seems thus prematurate to conclude that the difference between our numerical results and the ε-expansion is significant. This issue can only be settled by using approximate algorithms which would allow one to perform a much larger number of iterations.

7. CONCLUSIONS

We have studied the magnetic susceptibility per site of the hierarchical Ising model for $\varepsilon = 1$ and 0. We found that the β dependence of this quantity could be fitted quite accurately with a simple power law. Our analysis of the high-temperature expansion proved that this power law is not exact. This approximate law implies a simple approximate formula for the coefficients of the high-temperature expansion, and, more importantly, approximate relations among the coefficients themselves. We found that the simplest of these relations is satisfied with relative errors less then 2% in the two cases considered. These approximate relations mean that special combinations of high-temperature graphs are quite small. We are presently trying to get a systematic understanding of this situation. This task is made difficult by the fact that, due to the nonlocality of the hierarchical model, the calculation of the high-temperature coefficients is as difficult as the calculation of Feynman diagrams. For instance, the calculation of the fifth coefficient of the susceptibility has a difficulty comparable to the calculation of two-loop Feynman diagrams.

We insist on the fact that we have made no approximations in our numerical calculations and that the results presented are exact up to

roundoff errors. We have analyzed these errors and found that they do not affect our conclusions. The fact that we were able to reproduce accurately well-understood analytical results such as the scaling laws at the critical temperature and the high-temperature behavior of the susceptibility seems to rule out errors in implementing Eq. (2.4) numerically.

We have not reproduced accurately the value of the critical exponent γ calculated with the epsilon expansion, even when the fit of the power law is restricted to intervals closer to β_c. Even after 17 iterations of (2.4), which corresponds to the integration of 131,072 spins, the power behavior of the susceptibility stays closer to the high-temperature behavior than to the predictions of the RG analysis. The most plausible explanation is provided by the RG analysis itself, which suggests that we need more iterations. This can only be achieved at the present time by using approximate algorithms, which could be tested using the exact results presented here. Another possibility that might be considered is the existence of another fixed point with different critical indices; however, we are not aware of any result suggesting (or ruling out) this situation.

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